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# On the Ferromagnetic Ising Model in Noninteger Spatial Dimension

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Using the necessary and sufficient conditions in terms of the high-field series coefficients that the Yang–Lee theorem holds, we prove rigorously by counterexample that it cannot be extended to general noninteger dimension, when such models are defined by the "natural" analytic continuation.

**KEY WORDS:** Ising model; Yang–Lee theorem; dimensional analytic continuation; series expansions methods.

## 1. INTRODUCTION AND SUMMARY

Over the past few decades, the idea of using the spatial dimension, $d$, as a continuous variable to facilitate the study of the spin-1/2 Ising model (among others) has gained a considerable popularity. For example, Fisher and Gaunt[1] computed a series in $(1/d)$ for the critical temperature. Further studies were made by Abe,[2] again based on $(1/d)$ expansions. An additional line of work using this idea was the powerful computational device[3] in the renormalization group theory of critical phenomena[4] in which the critical exponents were expanded in powers of 4-$d$. These various techniques depend implicitly on the idea that the sought quantities are analytic in $d$ over a suitable region. Studies have been carried out which show that the $(1/d)$ expansions are likely to be at best asymptotic,[5] and that the (4-$d$) expansions are at best asymptotic.[6] Nevertheless, the possibility remains that these series are still summable.

Surprisingly little attention has been paid, to date, to the nature of these models in noninteger dimensional spaces. Wilson[7] has stated that these spaces are embeddable in an infinite-dimensional space, however, as we shall see, not in the usual sense. In this paper we will investigate some

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aspects of the nature of the spin-1/2 Ising model in noninteger dimensional space. The famous Yang–Lee theorem[8] on the location of zeros of the partition function has been proven to hold over a wide class of Ising and Ising-like ferromagnetic models.[9] In the second section we discuss how this theorem implies a family of inequalities which are satisfied by the coefficients in the high-magnetic field series expansion. In order to apply these inequalities we derive in the third section the appropriate expansion coefficients through eighth order in general dimension. To make connection with the analytic continuation in $d$ used in the (4-$d$) expansions, which at first sight appears to be a different one, we explore in Section 4 the relation between it and the "natural" continuation used in Section 3 and we find that they are in fact the same. In our fifth section, we give a detailed report of the regions of validity and failure which we have found for the inequalities which follow from the Yang–Lee theorem. It appears from the behavior of our results with the number of series coefficients used that, given enough series terms, the failure of the Yang–Lee theorem would be universal, except for (i) arbitrary dimension and infinite temperature, and (ii) arbitrary temperature and integer dimension. In our sixth section we look at the radius of convergence of the high-field series and find that the nature of failure of the Yang–Lee theorem is such that the radius of convergence decreases well below the Yang–Lee theorem lower bound of unity. Finally, in the last section we present a speculative picture of the Ising model in noninteger dimension, in which the critical point is thought of as a spinodal point and the phase-boundaries beyond which it lies have vanishing singularities in integer dimensions.

## 2. CONSEQUENCES OF THE YANG–LEE THEOREM

In this section we review some consequences of the Yang–Lee theorem which can be explicitly tested in low-order perturbation theory. The Yang–Lee theorem[8] states that if we have a partition function,

$$Z = \int \cdots \int \exp\left[\sum_{i,j=1}^{n} J_{ij}\phi_i\phi_j + \sum_{i=1}^{n} h_i\phi_i\right] \prod_{i=1}^{n} d\nu_i(\phi_i) \tag{2.1}$$

where $J_{ij} \geqslant 0$, then $Z \neq 0$ whenever $\mathrm{Re}\, h_i > 0$ or $\mathrm{Re}\, h_i < 0$ for all $i$, and $d\nu$ is a suitable measure. When all $h_i = h$, $n$ is finite, and $d\nu = \frac{1}{2}[\delta(\phi + 1) + \delta(\phi - 1)]$ (spin-1/2 Ising model) then $(e^{nh}Z)$ is a polynomial in $\mu = e^{-2h}$, and consequently has all its zeros on the unit circle in the complex $\mu$-plane. One of the most general proofs of this theorem is due to Lieb and Sokal,[9] where they show that the requirement,

$$\int e^{h\phi}\, d\nu_i(\phi) \neq 0, \qquad \mathrm{Re}\, h \neq 0, \quad \text{all } i \tag{2.2}$$

is sufficient to prove the Yang–Lee theorem. In this paper we will be concerned with a model which can be defined for any $h$, $|h| < \infty$. That means for us that both the integrals (2.1) and (2.2) are absolutely convergent for any such $h$. In order for (2.1) to converge with a nonzero $J_{ij}$, $d\nu$ must decay for large $\phi$ at least as fast as $\exp[-(A\phi)^2]$ for some $A$. Thus the rate of increase of $Z$ can be no faster than $\exp[nBh^2]$ for some $B$. According to Hadamard's factorization theorem[10] we can write

$$Z(h) = e^{nsh+nth^2} \prod_{j=1}^{\infty} \left\{ \left(1 - \frac{h}{h_j}\right) \exp\left[ \frac{h}{h_j} + \frac{1}{2}\left(\frac{h}{h_j}\right)^2 \right] \right\} \tag{2.3}$$

For this general class of models we can follow the procedures of Baker[11] and Bessis *et al.*[12] We have directly from (2.3)

$$\frac{1}{n} \ln Z(h) = sh + th^2 + 2\int_0^G d\rho_n(g)\left[\ln(1 + g^2h^2) - g^2h^2\right] \tag{2.4}$$

where $d\rho_n$ is a nonnegative Stieltjes measure which gives the contributions of each zero $h_j$ appearing in (2.3), and $G < \infty$ as $Z(0) \neq 0$. Use has been made in writing (2.4) of the fact that (2.1) is real for real $h$ so the zeros of (2.3) appear in pairs, $h_j$ and $-h_j$. The magnetization per spin can now be written as

$$I(h) = \frac{1}{n} \frac{\partial \ln Z(h)}{\partial h} = 2th - 4h^3 \int_0^G d\rho_n(g) \frac{g^4}{1 + g^2h^2} \tag{2.5}$$

and we have set $s = 0$ because $I(0) = 0$ for the temperature higher than the critical temperature which is the case we wish to consider.

The next step is to introduce a mapping of the $h$ plane so that we can put $I$, or something closely related, in the form of a series of Stieltjes.[13] In the applications (spin-1/2 Ising model) we wish to consider, we will know the behavior of $I$ as a power series in $\mu$, and at least in integer dimensions this power series is known to converge in any compact subset of $|\mu| < 1$. The order of $Z$ as a function of $\mu$, by the bound on the growth in $h$ obtained above, is zero. Thus Hadamard's factorization theorem gives

$$e^{nh} Z(\mu) = \prod_{i=1}^{n} (1 - \mu/\mu_i) \tag{2.6}$$

where $|\mu_i| = 1$, and $Z(1) \neq 0$. The analog of (2.4) is

$$\frac{1}{n} \ln Z(\mu) = -h + \int_{\Theta_0}^{\pi} \ln(\mu^2 - 2\mu\cos\Theta + 1)\, dg(\Theta) \tag{2.7}$$

where $g(\Theta)$ is a positive measure since it is the density of zeros on the unit circle $V$ and $\Theta_0 > 0$, at least for finite $n$, as $Z(1) \neq 0$. The intensity of magnetization is

$$I(\mu) = \frac{1}{n} \frac{\partial \ln Z(h)}{\partial h} = 2(1 - \mu^2) \int_{\Theta_0}^{\pi} \frac{dg(\Theta)}{1 - 2\mu\cos\Theta + \mu^2} \tag{2.8}$$

We use the facts that there are exactly $n$ zeros and that if $\mu_i$ is a zero so is $\mu_i^*$. In the limit as $n \to \infty$, we have the normalization

$$\int_{\Theta_0}^{\pi} dg(\Theta) = \frac{1}{2} \tag{2.9}$$

In this form of $I(\mu)$ we make the substitution,

$$\begin{aligned} v &= \frac{4\mu}{(1+\mu)^2} = \operatorname{sech}^2 h = 1 - \tanh^2 h \\ \mu &= \left[2 - v - 2(1-v)^{1/2}\right]/v \end{aligned} \tag{2.10}$$

which yields

$$I(v) = 2(1-v)^{1/2} \int_{\Theta_0}^{\pi} \frac{dg(\Theta)}{1 - v\cos^2(\Theta/2)} \tag{2.11}$$

Thus as $(1-v)^{1/2} = \tanh h$,

$$\frac{I(v)}{\tanh h} = \int_0^{\cos^2(\Theta_0/2)} \frac{d\phi(w)}{1 - vw} \tag{2.12}$$

which is of the form of a series of Stieltjes of $-v$ with a radius of convergence greater than or equal to unity. We remark that as long as we retain the Taylor series in $\mu$ property, the general case (2.3)–(2.5) can be reduced to the form

$$\frac{I(v)}{\tanh h} = C - v\int_0^{D} \frac{d\Psi(w)}{1 - vw} \tag{2.13}$$

where $D \leqslant 1$, and $C$ is a constant.

A series of Stieltjes is any function which can be represented as

$$f(x) = \sum_{j=0}^{\infty} f_j(-x)^j = \int_0^{R^{-1}} \frac{d\omega(w)}{1 + xw} \tag{2.14}$$

where, clearly $R$ is a lower bound on the radius of convergence, and $R = 0$ is allowed. Many properties of these functions are described in the literature.[13] For our purposes, the following results will suffice: it is necessary and sufficient that

$$D(m,n) = \det \begin{vmatrix} f_m, & f_{m+1}, \ldots, & f_{m+n} \\ \vdots & \vdots \quad \ddots & \vdots \\ f_{m+n}, & f_{m+n+1}, \ldots, & f_{m+2n} \end{vmatrix} > 0, \qquad \begin{aligned} m &= 0, 1 \\ n &= 0, 1, 2, \ldots \end{aligned} \tag{2.15}$$

hold for $f(x)$ to be of form (2.14), $0 \leqslant R \leqslant \infty$. Furthermore, if we restrict

$R \geqslant 1$, and define

$$\Delta^n f_m = \Delta^{n-1} f_m - \Delta^{n-1} f_{m+1}, \qquad \Delta^0 f_m = f_m \tag{2.16}$$

then it is also necessary and sufficient that[14] the sequence $\{f_m\}$ is totally monotone, i.e.,

$$\Delta^n f_m \geqslant 0 \qquad \text{all } m, n \tag{2.17}$$

In the case that only $f_0, \ldots, f_p$ are available, it suffices to check

$$\Delta^p f_0 \geqslant 0, \qquad \Delta^{p-1} f_1 \geqslant 0, \ldots, \Delta^0 f_p \geqslant 0 \tag{2.18}$$

If we find a violation of (2.15) or (2.18) in the series coefficients of (2.12), then it cannot be a series of Stieltjes and hence the Yang–Lee theorem will necessarily fail for that case.

## 3. THE HIGH-FIELD EXPANSION IN GENERAL DIMENSION

We need for this work, the expansion of the magnetization $I(\mu, u)$ for the spin-1/2 Ising model in arbitrary (integer for now) dimension in powers of $\mu$ for fixed $u$ where we pick

$$\mu = e^{-2h}, \qquad u = e^{-4K} \tag{3.1}$$

and the Hamiltonian is

$$Z = \sum_{\{\nu_i = \pm 1\}} \exp\left[ K \sum \nu_i \nu_j + h \sum_i \nu_i \right] \tag{3.2}$$

Fortunately, most of the work for this project has been done. We can deduce from (2.7) of Baker[15] the expansion for general $d$ through $\beta_7$ for the hypercubic lattice system

$$\ln Z = x\left(1 - \sum_{k \geqslant 1} \frac{k}{k+1} \beta_k x^k\right) \tag{3.3}$$

where

$$x = \tfrac{1}{2}(1 - \langle \nu_i \rangle) \tag{3.4}$$

By standard theory,[16] the coefficients in (3.3) can be used to give

$$z \equiv u^d \mu = x \exp\left[ - \sum_{k=1}^{\infty} \beta_k x^k \right] \tag{3.5}$$

This equation can be reverted to give

$$I = \langle \nu_i \rangle = 1 - 2x \tag{3.6}$$

as a series in $z$ with coefficients which depend on $u$. It is convenient in this work to use Mayer's $f$,

$$f = u^{-1} - 1 \tag{3.7}$$

as a variable. We find

$$
\begin{aligned}
\beta_1 &= -1 + 2df \\
\beta_2 &= -\frac{1}{2} - 3df^2 \\
\beta_3 &= -\frac{1}{3} + 2df^2 + \frac{20}{3}df^3 + 4\binom{d}{2}f^4 \\
\beta_4 &= -\frac{1}{4} - 10df^3 - \left[\frac{35}{2}d + 20\binom{d}{2}\right]f^4 - 40\binom{d}{2}f^5 \qquad (3.8) \\
\beta_5 &= -\frac{1}{5} + 4df^3 + \left[42d + 36\binom{d}{2}\right]f^4 + \left[\frac{252}{5}d + 240\binom{d}{2}\right]f^5 \\
&\quad + \left[276\binom{d}{2} + 96\binom{d}{3}\right]f^6 + \left[12\binom{d}{2} + 72\binom{d}{3}\right]f^7, \\
\beta_6 &= -\frac{1}{6} - \left[35d + 28\binom{d}{2}\right]f^4 - \left[168d + 560\binom{d}{2}\right]f^5 \\
&\quad - \left[154d + 1932\binom{d}{2} + 672\binom{d}{3}\right]f^6 - \left[1764\binom{d}{2} + 2184\binom{d}{3}\right]f^7 \\
&\quad - \left[238\binom{d}{2} + 1176\binom{d}{3}\right]f^8 + 56\binom{d}{3}f^9 \\
\beta_7 &= -\frac{1}{7} + \left[10d + 8\binom{d}{2}\right]f^4 + \left[216d + 640\binom{d}{2}\right]f^5 \\
&\quad + \left[660d + 5520\binom{d}{2} + 1920\binom{d}{3}\right]f^6 \\
&\quad + \left[490\frac{2}{7}d + 13680\binom{d}{2} + 14880\binom{d}{3}\right]f^7 \\
&\quad + \left[11748\binom{d}{2} + 27696\binom{d}{3} + 5184\binom{d}{4}\right]f^8 \\
&\quad + \left[2832\binom{d}{2} + 13120\binom{d}{3} + 6400\binom{d}{4}\right]f^9 \\
&\quad + \left[48\binom{d}{2} + 144\binom{d}{3} + 2560\binom{d}{4}\right]f^{10} + 96\binom{d}{3}f^{11} + 8\binom{d}{3}f^{12}
\end{aligned}
$$

where $\binom{d}{n}$ is the standard binomial coefficient. We remark that to compute $\beta_7$, in addition to the $d_{n,s}$ given in (2.10) of Baker[15] we have required $d_{10,2}$, $d_{11,3}$, and $d_{12,4}$ which we were able to supply without the need for further lattice embedding data.

Without going into detail[17] we give the usual revision for (3.5) as

$$x = \sum_{j \geq 1} jb_j z^j \qquad (3.9)$$

where

$$jb_j = \frac{1}{j} \sum_{n_k} \prod_{k=1}^{j-1} \frac{(j\beta_k)^{n_k}}{n_k!}, \qquad \sum_{k=1}^{j-1} kn_k = j - 1 \tag{3.10}$$

As an illustration, the first few values of (3.10) are

$$\begin{aligned} 1 \cdot b_1 &= 1 \\ 2 \cdot b_2 &= \beta_1 \\ 3b_3 &= \beta_2 + \tfrac{3}{2}\beta_1^2 \\ 4b_4 &= \beta_3 + 4\beta_1\beta_2 + \tfrac{8}{3}\beta_1^3 \\ &\cdots \end{aligned} \tag{3.11}$$

It is interesting to note from the structure of the coefficients that divergence can occur as $T \to 0$, i.e., $u \to 0$ for noninteger values of $d$. To see this result, we consider the $D$-dimensional unit hypercube. It has $2^D$ vertices and $D2^{D-1}$ edges and so will first appear $\binom{d}{D}$ times as a term in $\beta_{2^D}$ and contribute an $f^{D2^{D-1}}$ to this irreducible cluster sum. From (3.9) and (3.5) we will have, overall, for $u \to 0$

$$\binom{d}{D} z^{2^D} f^{D2^{D-1}} \propto \binom{d}{D} \mu^{2^D} u^{(2d-D)2^{D-1}} \tag{3.12}$$

Of course, for *integer* $d < D$ the coefficient in (3.12) vanishes so no trouble arises. However, if $d < \frac{1}{2}D$ and not an integer, then (3.12) is a divergent contribution as $u \to 0$. It is evident from $f_0$ finite that this result is a violation of (2.17) and so would imply that the Yang–Lee theorem must fail at least by about the $2^{2d}$th term for any noninteger $d$. In fact this result, as we will see later, is just illustrative of much more extensive failures.

We remark that (3.12) is also suggestive in regard to the radius of convergence of the $\mu$ series, though without an analysis of possible cancellations it is not possible to do better. The dominant contributions to the coefficient of $\mu^{2D}$ for small enough $u$ will be highly compact clusters of which the hypercube is an example. For $d$ a fixed noninteger, and $D$ very large $\binom{d}{D} \propto d!\,D^{-d}$ with a coefficient which is related to the deviation of $d$ from an integer. The Cauchy $n$th root test applied to (3.12) yields $u^{D/2-d}$ as an estimate of the radius of convergence of the $\mu$ series which goes to zero as $D \to \infty$ for any $u < 1$! The rate here is logarithmically slow in order, so it might be very difficult to observe numerically.

## 4. DIMENSIONAL CONTINUATION

In the previous section we saw that the coefficients, which are valid in arbitrary integer dimension, only contain $d$ in simple polynomial expressions. The natural dimensional continuation which suggests itself is just to use those same polynomial expressions in arbitrary noninteger dimension as well. Manifestly this continuation is not unique as $P(d)$ and $P(d) + \sin \pi d$ are both analytic in $d$ and agree for $d$ integer. In this section we investigate some other methods of analytic continuation which have been used to see how they relate to the polynomial method.

First there is the fractal method of approaching noninteger dimension. In the studies[18] that we know of, the fractal lattice is defined as a limiting process, embedded in some Euclidean space of dimension $d_E > d$, which proceeds through a sequence of finite systems of sites and bonds. In each case, every member of this limiting process is an Ising model within the scope of Section 2. Hence, by standard limiting arguments, the Yang–Lee theorem must hold for these fractal models. To anticipate the results of subsequent sections, since the Yang–Lee theorem fails for the polynomial dimensional continuation in noninteger dimension, fractal continuation is definitely a different continuation.

In the $\epsilon$-expansion calculations of the renormalization group theory of critical phenomena, $\epsilon = 4 - d$ where $d$ is the spatial dimension. Here dimensional continuation is based on the analytic continuation of two types of integrals.[19,20] First

$$I_1 = \frac{1}{(2\pi)^d} \int d^d\mathbf{k}\, f(k^2) = K_d \int_0^\infty k^{d-1} f(k^2)\, dk \tag{4.1}$$

where

$$K_d = \left[2^{d-1}\pi^{d/2}\Gamma\left(\tfrac{1}{2}d\right)\right]^{-1} \tag{4.2}$$

with $\Gamma(n)$ the gamma function. Second,

$$\begin{aligned} I_2 &= \frac{1}{(2\pi)^d} \int d^d\mathbf{k}\, f(k^2, \mathbf{k}\cdot\mathbf{k}_1) \\ &= \frac{1}{2\pi} K_{d-1} \int_0^\infty dk \int_0^\pi d\Theta\, k^{d-1} (\sin\Theta)^{d-2} f(k^2, k_1 k \cos\Theta) \end{aligned} \tag{4.3}$$

In fact, examination of the integrals to which (4.1) and (4.3) are applied shows that they are always of the form

$$I = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} \cdots \int \prod_{j=1}^{M} d^d\mathbf{k}_j \prod_{i=1}^{N} \frac{1}{D_i} \tag{4.4}$$

where the $D_i$ are positive-definite, quadratic forms in the $\mathbf{k}_j$. These quadratic forms are composed only of vector inner products (no cross-products, etc.) and hence decompose as

$$D_i = \sum_{\tau=1}^{d} \mathfrak{D}_i(\mathbf{k}_1 \cdot \mathbf{e}_\tau, \ldots, \mathbf{k}_m \cdot \mathbf{e}_\tau) \tag{4.5}$$

where $\mathbf{e}_\tau$ are the $d$ unit vectors in the coordinate directions. Irrespective of what we plan to do about dimensional continuation, we can use the identity,

$$\frac{1}{a} = \int_0^\infty e^{-ax}\, dx, \qquad a > 0 \tag{4.6}$$

to rewrite (4.4) as

$$I = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} \cdots \int \prod_{j=1}^{M} d^d\mathbf{k}_j \int_0^\infty \cdots \int \prod_{i=1}^{N} dx_i \exp\left( - \sum_{i=1}^{N} x_i D_i \right) \tag{4.7}$$

At this point we remark that form (4.7) is now adapted to the "product method" of dimensional continuation. For this we interchange the order of the $x$ and $\mathbf{k}$ integrations and obtain, by (4.5)

$$I = \frac{1}{(2\pi)^d} \int_0^\infty \cdots \int \prod_{i=1}^{N} dx_i \left\{ \int_{-\infty}^{+\infty} \cdots \int \prod_{j=1}^{M} d(\mathbf{k}_j \cdot \mathbf{e}_1) \right.$$

$$\left. \times \exp\left[ - \sum_{i=1}^{N} x_i \mathfrak{D}_i(\mathbf{k}_1 \cdot \mathbf{e}_1, \ldots, \mathbf{k}_m \cdot \mathbf{e}_1) \right] \right\}^d \tag{4.8}$$

which extends in an obvious way to nonintegral dimension. The Wilson prescription (4.1)–(4.3) is actually equivalent. To see this we first note that (4.1) follows from (4.3) if $k_1 = 0$ as

$$\int_0^\pi d\Theta\, (\sin\Theta)^{d-2} = \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}(d-1))}{\Gamma(\frac{1}{2}d)} \tag{4.9}$$

by Pierce's Tables[21] #484. Then if we begin to integrate (4.7) by the Wilson method we get, concentrating on $\mathbf{k}_1$ first,

$$I = \frac{1}{(2\pi)^d} \int_0^\infty \cdots \int \prod_{i=1}^{N} dx_i \int_{-\infty}^{+\infty} \cdots \int \prod_{j=1}^{M} d^d\mathbf{k}_j \exp(-A\mathbf{k}_1^2 + B\mathbf{k}_1 \cdot \mathbf{q} + C) \tag{4.10}$$

$$= \frac{1}{2\pi} K_{d-1} \int_0^\infty \cdots \int \prod_{i=1}^{N} dx_i \int_{-\infty}^{+\infty} \cdots \int \prod_{j=2}^{M} d^d\mathbf{k}_j$$

$$\times \int_0^\infty dk_1 \int_0^\pi d\Theta\, k_1^{d-1} (\sin\Theta)^{d-2} \exp(-Ak_1^2 + Bk_1 q \cos\Theta + C) \tag{4.11}$$

where $A$ and $B$ depend on the $x_i$ only and $C$ depends on the $x_i$ and $\mathbf{k}_j$,

$j = 2, \ldots, M$. If we compute the $\Theta$ integral[22] we obtain, valid for noninteger $d$,

$$I = \frac{1}{2\pi} K_{d-1} \pi^{1/2} \Gamma\left(\tfrac{1}{2}(d-1)\right) \int_0^{\infty} \cdots \int \prod_{i=1}^{N} dx_i$$
$$\times \int_{-\infty}^{+\infty} \cdots \int \prod_{j=2}^{M} d^d \mathbf{k}_j \int_0^{\infty} dk_1 k_1^{d/2}$$
$$\times \frac{e^C}{\left(\frac{1}{2}Bq\right)^{d/2-1}} I_{d/2-1}(Bk_1 q) \exp(-Ak_1^2) \tag{4.12}$$

Now performing the integral over $k_1$ we get[23] (again valid for nonintegral $d$)

$$I = \int_0^{\infty} \cdots \int \prod_{i=1}^{N} dx_i \int_{\infty}^{+\infty} \cdots \int \prod_{j=2}^{M} d^d \mathbf{k}_j \frac{\exp(B^2q^2/4A + C)}{\left[2(\pi A)^{1/2}\right]^d} \tag{4.13}$$

which we can directly verify is the same result as is obtained by the direct use of (4.8) and doing the integral over $\mathbf{k}_1$. However, the form (4.13) is, in so far as the integrals over the $\mathbf{k}$'s are concerned, the same as (4.7) except there is one less integral to do. Thus by $M$ repeated applications of the above steps we concluded that the Wilson method of dimensional continuation is equivalent to the product method.

We next illustrate that the product method is equivalent to the polynomial method. It is convenient to do so for the lattice-cutoff commutator. It is

$$C = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int \prod_{i=1}^{d} d\Theta_i \left(1 - 2K \sum_{i=1}^{d} \cos\Theta_i\right)^{-1} \tag{4.14}$$

By the product method[24] Eq. (4.14) becomes immediately, as outlined above,[22]

$$C = \int_0^{\infty} e^{-x} \left[I_0(2Kx)\right]^d dx \tag{4.15}$$

which becomes, on expansion in $K$ followed by integration term by term,

$$C = 1 + 2dK^2 + 6(2d^2 - d)K^4 + 20(6d^3 - 9d^2 + 4d)K^6 + \cdots \tag{4.16}$$

The product method always generates polynomial coefficients through the application of the bionomial expansion theorem to the expanded form of $[\ ]^d$ in (4.8). Thus the Wilson prescription is actually equivalent to the polynomial continuation which we adopt in this study.

## 5. RIGOROUS NUMERICAL RESULTS

In Section 2 we saw that, as a consequence of the Yang–Lee theorem, the magnetization per spin, $I$, in a spin-1/2 Ising model gives rise to a function $I(v)/(1-v)^{1/2}$ which, for fixed $u$, has the form of a series of Stieltjes [see Eq. (2.12)] with a radius of convergence greater than or equal to unity. We have constructed (along the lines of Sections 3 and 4) the power series in $v$ (through order $v^8$) representing this function for general dimension $d$ (integer and noninteger) and fixed $u$, and we have checked to see whether these power series were indeed appropriate series of Stieltjes. The important checks were provided by the properties indicated in Eqs. (2.15) and (2.18).

We find that the power series are *in all probability not series of Stieltjes for any noninteger* $d$ *and any* $u \neq 1$. Our reasons for this claim are the following:

(i) It can be shown analytically that

$$D(0,1) = \det\begin{vmatrix} f_0 & f_1 \\ f_1 & f_2 \end{vmatrix} \tag{5.1}$$

$$= \frac{u^{2d}}{8}\left\{\left[\frac{1}{u^{2d}} - 1\right] - (2d)\left[\frac{1}{u} - 1\right]\right\} \tag{5.2}$$

Using the simple fact that $u = e^{\ln u}$, we have the following:

$$D(0,1) = \frac{u^{2d}}{8}\left\{\left[\sum_{k=0}^{\infty} \frac{(2d)^k[\ln(1/u)]^k}{k!} - 1\right]\right.$$
$$\left. - (2d)\left[\sum_{k=0}^{\infty} \frac{[\ln(1/u)]^k}{k!} - 1\right]\right\} \tag{5.3}$$

and so

$$D(0,1) = \frac{u^{2d}}{8} \sum_{k=2}^{\infty} \left[(2d)^k - 2d\right] \frac{[\ln(1/u)]^k}{k!} \tag{5.4}$$

This power series has an infinite radius of convergence for finite $d$. Since for $0 < u < 1$ and $0 < d < 1/2$ each term in the series is negative, we conclude that

$$D(0,1) < 0 \qquad \text{for} \quad 0 < u < 1 \quad \text{and} \quad 0 < d < 1/2 \tag{5.5}$$

In a similar way, starting from

$$D(0,1) = \frac{u^{2d-1}}{8}\left\{\left[u^{1-2d} - u\right] - (2d)\left[1 - u\right]\right\} \tag{5.6}$$

one finds that

$$D(0,1) < 0 \quad \text{for} \quad 1 < u < \infty \quad \text{and} \quad 0 < d < 1/2 \tag{5.7}$$

*Results* (5.5) *and* (5.7) *clash with property* (2.15). Equation (5.7) is not particularly surprising, as $u > 1$ is the antiferromagnetic case. Interestingly, (5.5) and (5.7) correspond to one of the fundamental inequalities given by Beckenback and Bellman.[25]

(ii) For $d > 1/2$ and $u < 1$ we ran careful numerical checks of properties (2.15) and (2.18), and we find that one or both of them are *not* satisfied for practically any $u$ when $d \leqslant 1.3$ (except, of course, for $d = 1$).

The detailed results of our numerical checks are represented in Fig. 1. In the regions of $u$–$d$ space labeled by integer $j$ the first $j$ coefficients of the power series under study fail to satisfy at least one property of series of Stieltjes while in those same regions the first $(j-1)$ coefficients satisfy all properties of series of Stieltjes that we could think of. For example, in the part of Fig. 1 labeled 5 condition (2.18) fails for $f_0$, $f_1$, $f_2$, $f_3$, and $f_4$ while both condition (2.15) and condition (2.18) hold for $f_0$, $f_1$, $f_2$, and $f_3$.

(iii) From Fig. 1 it seems very likely that if we had more terms in the power series of interest, we would find more extended regions of $u$–$d$ space in which the power series fails to be a series of Stieltjes.

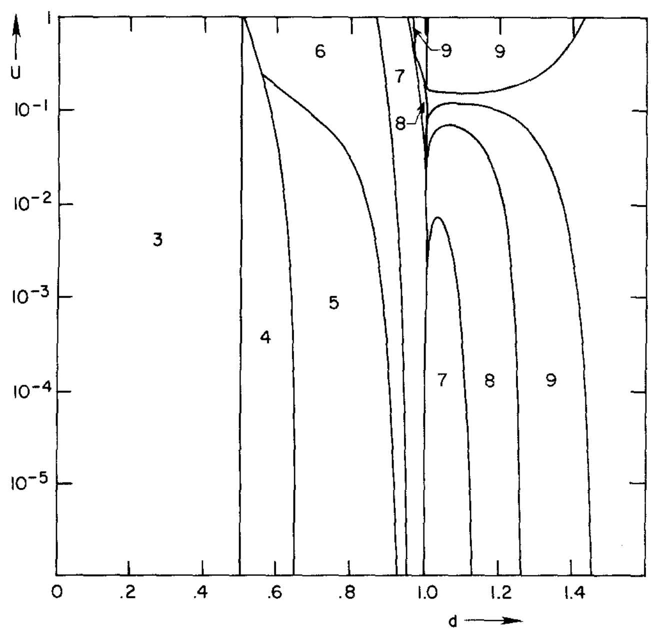


Fig. 1. Regions of the $u$–$d$ space in which $I(v)/(1-v)^{1/2}$ is found not to be a series of Stieltjes. In these regions the Yang–Lee theorem breaks down.

These results prove rigorously that the standard "nonfractal" continuation of the spin-1/2 Ising model to noninteger dimensions fails to preserve the Yang–Lee theorem.

It is interesting to note that for $d < 1$ the "determinantal" property (2.15) is the sensitive indicator, and it determines all boundaries in Fig. 1 except for the one between $j = 5$ and 6. On the other hand, for $d > 1$ and $u < 0.15$ the "finite difference" property (2.18) is the more sensitive indicator.

We have also checked the possibility that $I(v)/\tanh h$ will assume the form presented in Eq. (2.13): in other words, we studied the power series in $v$ representing

$$F = \frac{1}{v}\left[\frac{I(v)}{\tanh h} - C\right] \tag{5.8}$$

for different dimensions and $u < 1$. Using the properties (2.15) and (2.18), extensive numerical work led us to conclude that for $u < 1$ and noninteger dimensions the power series representing $F$ are also *not* series of Stieltjes. The detailed results of this work appear in Fig. 2, which is to be interpreted in a manner similar to Fig. 1.

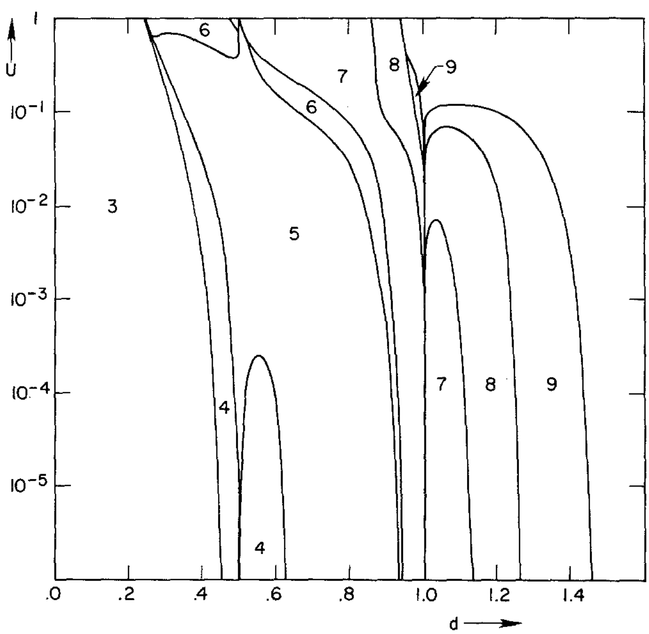


Fig. 2. Regions of the $u$–$d$ space in which $[I(v)/(1-v)^{1/2} - C]/v$ is found not to be a series of Stieltjes.

## 6. RADII OF CONVERGENCE

We have formed estimates of the radii of convergence for the power series in $v$ representing $I(v)/(1-v)^{1/2}$ for integer and noninteger $d$ and $u < 1$. These estimates are based on

(i) the "Cauchy method"

$$R \sim \frac{1}{\sqrt[n]{f_n}} \qquad \text{for } n \text{ large} \tag{6.1}$$

(ii) the location of the pole of the [4/4] Padé approximant for $I(v)/(1-v)^{1/2}$ closest to the origin in the $v$ plane.

In our work estimates based on (ii) were always smaller than those based on (i). We remark that if (2.15) were to hold for all $n$, but not necessarily (2.18), then (ii) would be a rigorous upper bound on the radius of convergence. As remarked in Section 5, for $d > 1$ and $u < 0.15$ such a case is possible (although we do not believe it to be likely), but in the other cases where (2.15) fails method (ii) is only an estimate.

For $1/2 < d < 1$ and $u \lesssim 0.1$ the radii of convergence can be said to be smaller than unity. The smaller $u$, the smaller the radius of convergence is:

$$\text{if} \quad u \simeq 0.01, \qquad R < 10^{-2}$$

and

$$\text{if} \quad u \simeq 0.0001, \qquad R < 10^{-5}$$

based on the Padé method. In these cases the Cauchy method gives $R < 0.6$ and $R < 0.05$, respectively.

For $d = 1$ and $u \lesssim 0.1$ the radius of convergence is only slightly greater than unity. At $d = 1$ a dramatic jump occurs in the radius of convergence from its values for $d < 1$. This jump is sharper for smaller $u$.

For $d > 1$ and $u \lesssim 0.1$ the radius of convergence first drops down to approximately the values it had for $d$ just less than unity. Then it starts increasing with $d$ (while $u$ is held fixed) and it becomes approximately unity as $d$ becomes $3/2$.

For $1/2 < d < 3/2$ and $u > 0.1$ the radius of convergence is estimated to be greater than unity. There the failure of the Yang–Lee theorem is due either to a loss of positivity of the density of zeros on the unit circle in the $\mu$ plane or to their wandering from that circle. For $u = 0.9$, for example, our estimates lead to radii of convergence $\sim 10$. In the regions of $u$–$d$ space where the power series of interest are not series of Stieltjes, the pole of the [4/4] Padé approximant nearest the origin in the $v$ plane is always accompanied very closely by a zero. When the series satisfy all the properties of a series of Stieltjes with radius of convergence unity (for example, when $d = 1$) the pole and zero separate significantly.

## 7. OPEN QUESTIONS

In this section we try to organize the results of the previous two sections into a coherent picture. It is helpful in this effort to take note of some additional known results of potential relevance. First it is easy to show for the Gaussian model that the critical temperature and the susceptibility critical index ($\gamma = 1$) are analytic functions of $d$. Beyond this Fisher and Gerber[5] have shown that for $2 < d < \infty$ the critical temperature of the spherical model is analytic in $d$, although the critical indices are only piecewise analytic,[26] $2 < d < 4$, $4 < d < \infty$. For the renormalization group theory of the $g_0 : \phi^4 :_d$ model Brezin *et al.*[6] have shown that the Borel transform of the $\epsilon$-expansion (see Section 4) has a finite radius of convergence. By the Borel transform we mean the following. Let

$$F(t) = \sum_{k=0}^{\infty} F_k t^k \tag{7.1}$$

be a formal power series. Define its Borel transform as

$$f(t) = \sum_{k=0}^{\infty} \frac{F_k t^k}{k!} \tag{7.2}$$

also as a formal power series. Then if the series (7.2) converges and $f(t)$ has an analytic continuation over the interval $0 \leqslant t < \infty$ which does not grow too fast at $t \to \infty$, we give the Borel sum of (7.1) as

$$F(t) = \int_0^{\infty} e^{-x} f(xt)\, dx \tag{7.3}$$

by use of the identity

$$k! = \int_0^{\infty} x^k e^{-x}\, dx \tag{7.4}$$

The proof that $f(t)$ has a finite radius of convergence is a necessary first step to establishing the Borel summability of a formal power series (7.1). Also for the $g_0 : \phi^4 :_d$ field theory t'Hooft and Veltman[27] in their study of dimensional renormalization theory have shown that the perturbation series in $g_0$ is analytic in $d$, term-by-term. Here Rivasseau and Speer[28] have again shown that the Borel transform has a finite radius of convergence, so at least the Borel transform is analytic in $d$. The $g_0 : \phi^4 :_d$ theories obey the Yang–Lee theorem[29] and so are appropriate, for insight purposes, to our discussion.

It appears from our numerical work that the radius of convergence of the $v$ series (and hence the $\mu$ series) decreases significantly below unity and possibly to zero for $d$ noninteger and for any $u < 1$. *Perhaps* the following physical idea is relevant to this problem. As we saw in Section 3, a $D$-dimensional hypercube, or for that matter an arbitrary portion of a $D$-dimensional hypercubic lattice is embeddable on *any* $d$, noninteger, dimensional lattice no matter how large $D$, nor how small $d$. Now on such a lattice fragment the energetically most favorable state is for antiferromagnetically aligned spins and, as we saw at (3.12), this state is not compensated by the $z^n$ factor and hence leads to a divergence of the series. Physically, we expect this effect to set in for $u$ less than its antiferromagnetic critical value, but

$$1 - u_c = O(1/D) \tag{7.5}$$

so it could well dominate for any $u < 1$ as $D$ can be indefinitely large. Bear

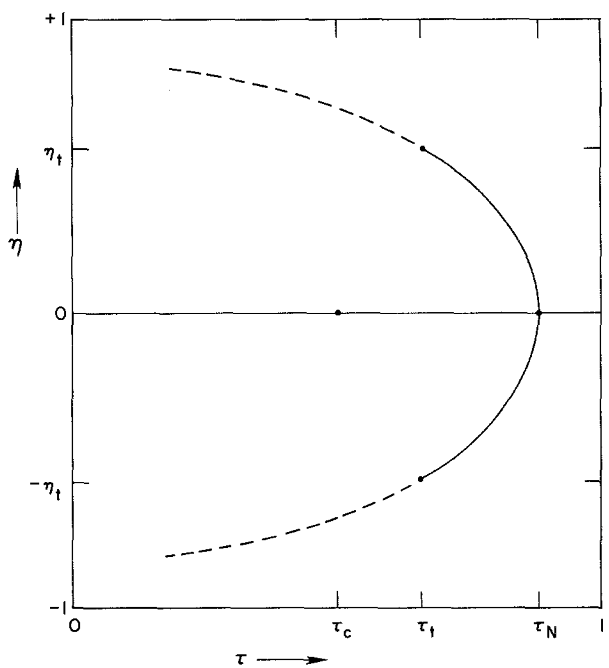


Fig. 3. Illustration of a simple antiferromagnetic phase diagram. $\tau = \tanh K$, $\eta = \tanh h$. $\tau_N$ is the Nèel point, $(\tau_t, \pm\eta_t)$ are the (possible) triple points. The solid curve is a line of second-order phase transitions and the dotted lines are lines of first-order phase transitions. $\tau_c$ labels the location of the ferromagnetic critical point in our picture.

in mind that this effect could be destroyed by cancellation. If it is not, then there will be a strong tendency for our system to undergo an antiferromagnetic phase transition in nonintegral dimensions at any finite temperature.

The usual critical point is in a state without staggered magnetization and in this picture would have to be considered a metastable state. In Fig. 3 we have shown a typical antiferromagnetic phase diagram with the usual ferromagnetic critical point indicated. If our suggested picture is correct the phase boundaries would lie on the upper, right, and lower boundaries of the figure, but the singularities on these boundaries would vanish for $d$ integer. The model for $d$ noninteger in current parlance would resemble an Ising model with competing interactions and a degree of frustration.

Except in integer dimensions the critical point computed by the $\epsilon$-expansion, for example, would be a spinodal point. Since spinodal points may be accessible (e.g., van der Waals gas), or may not be accessible (e.g., Ising model[30] or droplet models[31]) to series methods, it appears to be a nontrivial and interesting question whether the work of Brezin *et al.*[6] and Rivasseau and Speer[28] can be extended to demonstrate Borel summability of the $\epsilon$-expansion and the fixed, noninteger dimension $g_0$ expansion of the $g_0:\phi^4:_d$ model field theory. The high-temperature expansion must penetrate a line of second-order transitions and the high-field expansion probably a line of first-order transitions (See Fig. 3).

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